

EJERCICIOS DE DERIVADAS PARCIALES (para practicar)

1. Halla el gradiente y la hessiana en el punto $(0, 2)$ de la función $f(x, y) = x^3y - 2xy^2 + x^2 - 3y^2$.

Solución:

$$\text{Gradiente: } \begin{cases} f_x = 3x^2y - 2y^2 + 2x \\ f_y = x^3 - 4xy - 6y \end{cases} \Rightarrow \nabla f(0, 2) = (-8, -12).$$

Hessiana:

$$\begin{cases} f_{xx} = 6xy + 2 \\ f_{xy} = 3x^2 - 4y \end{cases}; \quad \begin{cases} f_{yx} = 3x^2 - 4y \\ f_{yy} = -4x - 6 \end{cases} \Rightarrow Hf(0, 2) = \begin{pmatrix} 2 & -8 \\ -8 & -6 \end{pmatrix}.$$

2. a) Halla el vector gradiente de la función $f(x, y) = \frac{x^2y - x^3}{x + y}$ en el punto $(1, -2)$.

b) Comprueba que es una función homogénea y que cumple el teorema de Euler.

Solución:

$$\text{a) } f_x(x, y) = \frac{(2xy - 3x^2) \cdot (x + y) - (x^2y - x^3) \cdot 1}{(x + y)^2} \Rightarrow f_x(x, y) = \frac{-2x^2y + 2xy^2 - 2x^3}{(x + y)^2}.$$

$$f_y(x, y) = \frac{x^2 \cdot (x + y) - (x^2y - x^3) \cdot 1}{(x + y)^2} \Rightarrow f_y(x, y) = \frac{2x^3}{(x + y)^2}$$

Gradiente:

$$f_x(1, -2) = \frac{-2 \cdot 1 \cdot (-2) + 2 \cdot 1 \cdot (-2)^2 - 2 \cdot 1^3}{(1 - 2)^2} = 10; \quad f_y(1, -2) = \frac{2}{(1 - 2)^2} = 2 \Rightarrow \nabla f(1, -2) = (10, 2).$$

$$\text{b) } f(tx, ty) = \frac{(tx)^2 \cdot ty - (tx)^3}{tx + ty} = \frac{t^3(x^2y - x^3)}{t(x + y)} = t^2 \cdot \frac{x^2y - x^3}{x + y} \rightarrow \text{Es homogénea de grado } k = 2.$$

Teorema de Euler: $xf_x + yf_y = kf$.

$$\begin{aligned} & \frac{x(-2x^2y + 2xy^2 - 2x^3)}{(x + y)^2} + \frac{y(2x^3)}{(x + y)^2} = \frac{-2x^3y + 2x^2y^2 - 2x^4 + 2yx^3}{(x + y)^2} = \frac{2x^2y^2 - 2x^4}{(x + y)^2} \\ & = \frac{2x^2(y^2 - x^2)}{(x + y)^2} = \frac{2x^2(y - x)(y + x)}{(x + y)^2} = \frac{2x^2(y - x)}{x + y} = 2 \cdot \frac{x^2y - x^3}{x + y}. \end{aligned}$$

3. Halla el gradiente y la hessiana en el punto $(-1, 1)$ de la función $f(x, y) = e^{x^2 - y}$.

Solución:

$$\text{Gradiente: } \begin{cases} f_x = 2xe^{x^2 - y} \\ f_y = -e^{x^2 - y} \end{cases} \Rightarrow \nabla f(-1, 1) = (-2, -1).$$

$$\text{Hessiana: } \begin{cases} f_{xx} = 2e^{x^2 - y} + 4x^2e^{x^2 - y} \\ f_{xy} = -2xe^{x^2 - y} \end{cases}; \quad \begin{cases} f_{yx} = -2xe^{x^2 - y} \\ f_{yy} = e^{x^2 - y} \end{cases} \Rightarrow Hf(-1, 1) = \begin{pmatrix} 6 & 2 \\ 2 & 1 \end{pmatrix}.$$

4. Halla en el punto (1, 1) el gradiente y la hessiana de $f(x, y) = \ln(2x - y)$.

Solución:

$$\text{Gradiente: } \begin{cases} f_x = \frac{2}{2x-y} \\ f_y = \frac{-1}{2x-y} \end{cases} \Rightarrow \nabla f(1, 1) = (2, -1).$$

$$\text{Hessiana: } \begin{cases} f_{xx} = \frac{-4}{(2x-y)^2} \\ f_{xy} = \frac{2}{(2x-y)^2} \end{cases}; \begin{cases} f_{yx} = \frac{2}{(2x-y)^2} \\ f_{yy} = \frac{-1}{(2x-y)^2} \end{cases} \Rightarrow Hf(1, 1) = \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}.$$

5. Halla el gradiente y la hessiana de la función $f(x, y) = \frac{x}{x-y}$ en el punto (1, -1).

Solución:

$$\text{Gradiente: } \begin{cases} f_x = \frac{1 \cdot (x-y) - x}{(x-y)^2} = -\frac{y}{(x-y)^2} \\ f_y = \frac{-x \cdot (-1)}{(x-y)^2} = \frac{x}{(x-y)^2} \end{cases} \Rightarrow \nabla f(1, -1) = \left(\frac{1}{4}, \frac{1}{4} \right).$$

Hessiana:

$$f_{xx} = \frac{-(-y \cdot 2(x-y))}{(x-y)^4} = \frac{2y}{(x-y)^3} \rightarrow f_{xx}(1, -1) = \frac{-2}{2^3} = -\frac{1}{4}.$$

$$f_{xy} = \frac{-(x-y)^2 - (-y \cdot 2(x-y) \cdot (-1))}{(x-y)^4} = \frac{-(x-y) - (-y \cdot 2 \cdot (-1))}{(x-y)^3} = \frac{-x-y}{(x-y)^3} \rightarrow f_{xy}(1, -1) = 0.$$

$$f_{yy} = \frac{-x \cdot (2(x-y) \cdot (-1))}{(x-y)^4} = \frac{2x}{(x-y)^3} \rightarrow f_{yy}(1, -1) = \frac{2}{2^3} = \frac{1}{4}.$$

$$Hf(1, -1) = \begin{pmatrix} -1/4 & 0 \\ 0 & 1/4 \end{pmatrix}.$$

6. Halla el gradiente y la hessiana de la función $f(x, y) = \sqrt{xy-2}$ en el punto (6, 1).

Solución:

$$\text{Gradiente: } \begin{cases} f_x = \frac{y}{2\sqrt{xy-2}} \\ f_y = \frac{x}{2\sqrt{xy-2}} \end{cases} \Rightarrow \nabla f(6, 1) = \left(\frac{1}{4}, \frac{6}{4} \right).$$

Hessiana:

$$f_{xx} = \frac{-y \cdot \frac{y}{2\sqrt{xy-2}}}{2(\sqrt{xy-2})^2} = -\frac{y^2}{4(xy-2)^{3/2}} \rightarrow f_{xx}(6,1) = -\frac{1^2}{4 \cdot 4^{3/2}} = -\frac{1}{32}.$$

$$f_{xy} = \frac{1 \cdot \sqrt{xy-2} - y \cdot \frac{y}{2\sqrt{xy-2}}}{2(\sqrt{xy-2})^2} = \frac{2(xy-2) - y^2}{2(\sqrt{xy-2})^2} = \frac{xy-4}{4(xy-2)^{3/2}} \rightarrow f_{xy}(6,1) = \frac{6 \cdot 1 - 4}{4 \cdot 4^{3/2}} = \frac{2}{32}.$$

$$f_{yy} = \frac{-x \cdot \frac{x}{2\sqrt{xy-2}}}{2(\sqrt{xy-2})^2} = -\frac{x^2}{4(xy-2)^{3/2}} \rightarrow f_{yy}(6,1) = -\frac{6^2}{4 \cdot 4^{3/2}} = -\frac{36}{32}.$$

$$Hf(6,1) = \begin{pmatrix} -1/32 & 2/32 \\ 2/32 & -36/32 \end{pmatrix}.$$

7. Halla el gradiente y la hessiana de la función $f(x, y) = 2x \cos(x - y)$ en el punto $(2, 2)$.

Solución:

$$\text{Gradiente: } \begin{cases} f_x = 2 \cos(x - y) - 2x \sin(x - y) \\ f_y = -2x \sin(x - y) \cdot (-1) = 2x \sin(x - y) \end{cases} \Rightarrow \nabla f(2, 2) = (2, 0).$$

Hessiana:

$$f_{xx} = -2 \sin(x - y) - 2 \sin(x - y) - 2x \cos(x - y) \rightarrow f_{xx}(2, 2) = -4.$$

$$f_{xy} = 2 \sin(x - y) + 2x \cos(x - y) \rightarrow f_{xy}(2, 2) = 4.$$

$$f_{yy} = -2x \cos(x - y) \rightarrow f_{yy}(2, 2) = -4.$$

$$Hf(2, 2) = \begin{pmatrix} -4 & 4 \\ 4 & -4 \end{pmatrix}.$$

8. Sea $f(x, y) = x^2 - 2y + \sin(xy)$. Halla su gradiente y su hessiana en el punto $(0, 1)$.

Solución:

$$\text{Gradiente: } \begin{cases} f_x(x, y) = 2x + y \cos(xy) \\ f_y(x, y) = -2 + x \cos(xy) \end{cases} \Rightarrow \nabla f(0, 1) = (1, -2)$$

Hessiana:

$$f_{xx} = 2 - y^2 \sin(xy) \rightarrow f_{xx}(0, 1) = 2.$$

$$f_{xy} = \cos(xy) - yx \sin(xy) \rightarrow f_{xy}(0, 1) = 1.$$

$$f_{yy} = -x^2 \sin(xy) \rightarrow f_{yy}(0, 1) = 0.$$

$$Hf(0, 1) = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}.$$

9. Halla el gradiente de la función $f(x, y) = x \sin(y^2) + \frac{e^{yx^2}}{2y+1}$ en el punto $(1, 0)$.

Solución:

$$f_x(x, y) = \sin(y^2) + \frac{2xye^{yx^2}}{2y+1} \rightarrow f_x(1, 0) = \sin 0 + \frac{0}{1} = 0;$$

$$f_y(x, y) = 2yx \cos(y^2) + \frac{x^2 e^{yx^2} \cdot (2y+1) - e^{yx^2} \cdot 2}{(2y+1)^2} \rightarrow f_y(1, 0) = 0 + \frac{1-2}{1} = -1.$$

Por tanto, $\nabla f(1, 0) = (0, -1)$.

10. Sea $f(x, y) = (2x - y^2)^3$. Halla su gradiente y hessiana en el punto $(1, -2)$.

Solución:

Gradiente:

$$f_x(x, y) = 3(2x - y^2)^2 \cdot 2 = 6(2x - y^2)^2; \quad f_y(x, y) = 3(2x - y^2)^2 \cdot (-2y) = -6y(2x - y^2)^2$$

$$\nabla f(1, -2) = (24, 48).$$

Hessiana:

$$f_{xx}(x, y) = 12(2x - y^2) \cdot 2 = 24(2x - y^2) \rightarrow f_{xx}(1, -2) = -48.$$

$$f_{xy}(x, y) = 12(2x - y^2) \cdot (-2y) \rightarrow f_{xy}(1, -2) = -96.$$

$$f_{yy}(x, y) = -6(2x - y^2)^2 - 6y \cdot 2(2x - y^2) \cdot (-2y) = -6(2x - y^2)^2 + 24y^2(2x - y^2) \rightarrow$$

$$f_{yy}(1, -2) = -216.$$

$$Hf(1, -2) = \begin{pmatrix} -48 & -96 \\ -96 & -216 \end{pmatrix}.$$